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Lorenz and Ginzburg-Landau equations for thermal convection in a high-porosity medium with heat source

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ABSTRACT

The paper presents an investigation of a weakly non-linear stability analysis of thermal convection in a porous medium using the Lorenz model. The Ginzburg-Landau model is then obtained from the Lorenz model using which an expression for Nusselt number, *Nu*, is obtained in closed form. Such a procedure of obtaining an analytical solution is reported for the first time in the literature. It is found that heat source enhances amount of heat transport whereas heat sink diminishes the same. The influence of the thermal and mechanical anisotropies on *Nu* is to oppose each other in the case of both heat source and heat sink. It is observed that the anisotropic effects are prevalent only for short time whereas heat source(sink) has a sustained influence. Several limiting cases are obtained from the present study.

1. Introduction

Understanding the convective instability in a fluid-saturated porous layer subjected to various additional effects has been given a great deal of effort due to its applications in a variety of engineering and geophysical problems, energy storage applications, oil recovery process in petroleum industry and others. One of the effective mechanisms of controlling convective instability is by maintaining a non-uniform temperature gradient across the fluid layer. Such a temperature gradient can be generated by uniformly distributed internal heat sources or by injecting the fluid at one of the boundaries and removal of the fluid at the other, called throughflow, or by uniform heating or cooling at the boundaries.

The study of natural convection in a porous medium has been understood and well documented in the works of Kaviany [18], Ingham and Pop [14], Vafai [46], Crolet [10], Straughan [39], Nield and Bejan [22] and Vadasz [43] and Rees [27].

Somerton et al. [37] studied natural convection in a volumetrically heated porous layer. Convection in a porous medium with internal heat source and variable gravity effects were analyzed by Rionero and Straughan [29]. Rao and Wang [26] studied natural convection in vertical porous enclosures with internal heat generation. Rees and Pop [28] investigated free convection induced by a vertical wavy surface with uniform heat flux in a porous medium. Coriolis effect on free convection in a long rotating porous box subject to uniform heat generation was investigated by Vadasz [41]. Thermal instability in an anisotropic porous medium with internal heat source and inclined temperature gradient was analyzed by Parthiban and Patil [24]. Free convection in a porous medium was analyzed by Vadasz [42]. Khalili and Shivakumara [19] investigated the onset of convection in a horizontal, isotropic porous layer including the effects of through-flow and a uniformly distributed internal heat generation for different types of hydrodynamic boundary conditions. Vadasz and Olek [45] studied the route to chaos for moderate Prandtl number convection in a porous layer heated from below. Onset of convection in a porous media with internal heat source and variable gravity was investigated by Herron [13]. Khalili et al. [20] studied the convective instabilities caused by a non-uniform temperature gradient due to vertical throughflow and internal heat generation in an anisotropic porous layer. Natural convection in a cavity with volumetric heat generation was investigated analytically by Joshi et al. [17] and Grosnan et al. [12]. Barletta et al. [4] conducted a linear stability analysis of the onset of convection in a porous layer induced by viscous dissipation. Influence of Darcy number on the onset of convection in a

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porous layer with uniform heat source was investigated by Nouri-Borujerdi et al. [23]. Cookey et al. [15] investigated the onset of thermal instability in a low Prandtl number fluid with internal heat source in a porous medium. Analytical prediction of the transition to chaos in Lorenz equations was studied by Vadasz [44] and Jawdat and Hashim [16]. Natural convection in a rotating anisotropic porous layer with internal heat generation was studied by Bhadauria et al. [6]. Bhadauria [5] investigated double diffusive convection in a saturated anisotropic porous layer with internal heat source.

Ginzburg-Landau equation is one of the most studied equations due to its application in explaining non-linear phenomenon. Siddheshwar [31] obtained a series solution for the Ginzburg-Landau equation with a time-periodic coefficient. Siddheshwar et al. [36] analyzed the heat transport in Bénard-Darcy convection with g-jitter and thermo-mechanical anisotropy in variable viscosity liquids using the Ginzburg-Landau model. Heat transport in a porous medium under g-jitter and internal heating effects was studied by Bhadauria et al. [7]. Alok et al. [8] extended the study of Bhadauria et al. [7] by considering the variable viscosity liquid. There are many other areas in which Ginzburg-Landau model appears (Barba-Ortega et al. [1–3].

Non-linear convection in porous media was extensively reviewed by Rudraiah et al. [30]. The study of finite amplitude convection [47], using a truncated Fourier representation, has gained momentum in recent years owing to its simplicity of approach in handling a non-linear problem. We note that the study of finite amplitude Rayleigh-Bénard convection in an anisotropic porous medium with internal heat generation by means of a minimal Fourier series representation does not seem to have been undertaken. Accordingly, we study this aspect in the paper and obtain Ginzburg-Landau equation by Lorenz model.

2. Mathematical formulation

An infinite extent horizontal porous layer of thickness d, whose lower and upper bounding planes are at $z = 0$ and $z = d$, respectively is considered. The porous layer is saturated by a viscous, Newtonian liquid. The upper and lower boundaries are maintained at constant temperatures T_0 and $T_0 + \Delta T$ $(\Delta T > 0)$ respectively. Further, the porous medium is supposed to be anisotropic, and Darcy law and the Oberbeck-Boussinesq approximation [25] are taken to be applicable. For mathematical tractability we confine ourselves to two-dimensional rolls so that all physical quantities are independent of y, a horizontal co-ordinate. The boundaries are assumed to be free and perfect conductors of heat. In this paper we assume the dynamic coefficient of viscosity, μ and thermal diffusivity, χ to be constants. Heat source is assumed to be temperature-dependent in the problem. The governing equations describing the Rayleigh-Bénard instability situation of a constant viscosity Newtonian fluid in an anisotropic porous medium are:

Conservation of mass

$$\nabla \cdot \vec{q} = 0, \tag{1}$$

Conservation of momentum

$$\rho_R \left[\frac{1}{\phi} \frac{\partial \vec{q}}{\partial t} + \frac{1}{\phi^2} (\vec{q} \cdot \nabla) \vec{q} \right] = -\nabla p + \rho \vec{g} - \mu_f \tilde{k} \cdot \vec{q} + \mu_p \nabla^2 \vec{q}, \tag{2}$$

Conservation of energy

$$\frac{\partial T}{\partial t} + \vec{q} \cdot \nabla T = \nabla \cdot [\tilde{\chi} \nabla T] + Q(T - T_0), \tag{3}$$

Equation of state

$$\rho = \rho_0 [1 - \beta(T - T_0)], \tag{4}$$

where $\vec{q} = (u, 0, w)$ is the velocity vector, p is the pressure, ϕ is the porosity of the porous medium, ρ and ρ_R are the density and reference density of the fluid, respectively, p is hydrodynamic pressure, $\vec{g} = (0, 0, -g)$ is the gravitational force, μ_f and μ_p are the dynamic viscosities of the fluid and the porous medium, respectively, $\tilde{k} = k_x^{-1} \hat{i}\hat{i} + k_z^{-1} \hat{k}\hat{k}$ is the permeability tensor, T is the temperature, $\tilde{\chi} = \chi_h \hat{i}\hat{i} + \chi_v \hat{k}\hat{k}$ is the thermal diffusivity tensor, $Q(T)$ is the temperature dependent heat source, t is the time and β is the coefficient of thermal expansion.

Taking the velocity, temperature and density fields in the quiescent basic state to be $q_b(z) = (0, 0), T_b(z)$ and $\rho_b(z)$, we obtain the quiescent state solution in the form:

$$\left. \begin{array}{l} q_b = (0, 0) \\ T_b = T_0 + \Delta T f\left(\frac{z}{d}\right) \\ \rho_b\left(\frac{z}{d}\right) = \rho_0 \left[1 - \beta \Delta T f\left(\frac{z}{d}\right)\right] \\ p_b\left(\frac{z}{d}\right) = -\int \rho_b\left(\frac{z}{d}\right) g d\left(\frac{z}{d}\right) + C_1 \end{array} \right\}, \tag{5}$$

where $f\left(\frac{z}{d}\right) = \frac{\sin\sqrt{R_1}\left(1 - \frac{z}{d}\right)}{\sin\sqrt{R_1}}$ and C_1 is the constant of integration. On the quiescent basic state we superimpose perturbation in the form:

$$\left. \begin{array}{l} q = q_b + q' \\ T = T_b\left(\frac{z}{d}\right) + T' \\ \rho = \rho_b\left(\frac{z}{d}\right) + \rho' \\ p = p_b\left(\frac{z}{d}\right) + p' \end{array} \right\}, \tag{6}$$

where the prime indicates a perturbed quantity. Since we consider only two-dimensional disturbances, we introduce stream function

$$u' = -\frac{\partial \psi'}{\partial z}, w' = \frac{\partial \psi'}{\partial x}. \tag{7}$$

These satisfy Eq. (1) in the perturbed state. Eliminating the pressure in Eq. (2), incorporating the quiescent state solution and non-dimensionalizing the resulting equations as well as Eq. (3) using the following definition

$$(X, Z) = \left(\frac{x}{d}, \frac{z}{d}\right), \tau = \frac{\chi_v}{d^2} t, \Psi = \frac{\psi'}{\chi_v}, \Theta = \frac{T'}{\Delta T}, \tag{8}$$

we obtain the dimensionless form of the vorticity and heat transport equations as

$$\frac{1}{Pr} \frac{\partial}{\partial \tau} (\nabla^2 \psi) = \Lambda \nabla^4 \psi - R_E \frac{\partial \Theta}{\partial X} - Da^{-1} \left(\frac{\partial^2 \Psi}{\partial X^2} + \frac{1}{\varepsilon} \frac{\partial^2 \Psi}{\partial Z^2} \right) + \frac{1}{Pr} J(\Psi, \nabla^2 \Psi), \tag{9}$$

$$\frac{\partial \Theta}{\partial \tau} = -\frac{\partial \Psi}{\partial X} \frac{df}{dZ} + \eta \frac{\partial^2 \Theta}{\partial X^2} + \frac{\partial^2 \Theta}{\partial Z^2} + R_I \Theta + J(\Psi, \Theta), \tag{10}$$

where $Pr = \frac{\nu}{\Phi k_v}$ is the Prandtl number, $Da^{-1} = \frac{d^2}{k_v}$ is the inverse Darcy number, $\epsilon = \frac{k_h}{k_v}, \eta = \frac{\kappa_h}{\kappa_v}$ are the mechanical and thermal anisotropy parameters, $\Lambda (= \frac{\mu_p}{\mu_f})$ is the Brinkmann number, $R_E = \frac{\beta g \Delta T d^3}{\nu \chi_v}$ is the external Rayleigh number, $R_I = \frac{Q d^2}{\chi_v}$ is the internal Rayleigh number, τ is the slow time scale, ψ is the dimensional stream function, Θ is the non-dimensional temperature and J is the Jacobian. The local acceleration term is retained in Eq. (9) as per the arguments put forth by Suthar et al. [40].

The non-dimensional parameters appearing in Eqs. (9) and (10) are as defined in the nomenclature.

Eqs. (9) and (10) are solved using the boundary conditions

$$\Psi = \frac{\partial^2 \Psi}{\partial Z^2} = \Theta = 0 \quad \text{at} \quad Z = 0, 1. \tag{11}$$

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In the next section, we discuss the linear stability analysis, which is of great utility in the local nonlinear stability analysis to be discussed further on.

3. Linear stability theory

In order to study the linear theory we consider the linear version of Eqs. (9) and (10) and assume the solutions to be periodic waves of the form [9]

$$\left.\begin{aligned}\Psi(X,Z,\tau) &= \tfrac{\sqrt{2}}{\pi^2\alpha_c}(k_2^2 - R_I)\Psi_0 e^{\sigma\tau}\sin\pi\alpha_c X\sin\pi Z\\ \Theta(X,Z,\tau) &= \tfrac{1}{r_E}\tfrac{4\sqrt{2}\pi}{(4\pi^2 - R_I)}\Theta_0 e^{\sigma\tau}\cos\pi\alpha_c X\sin\pi Z\end{aligned}\right\}, \tag{12}$$

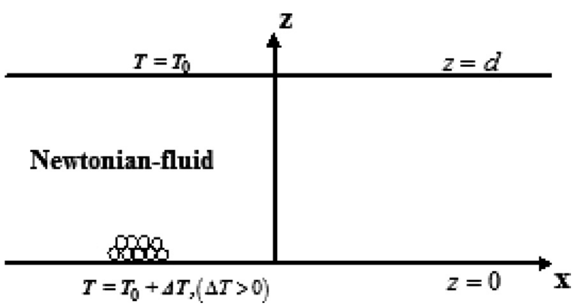

Fig. 1. Schematic of the flow configuration.

where

Ψ is the dimensionless stream function, σ is the growth rate and $r_E = \frac{R_E}{R_{Ec}}$ is the scaled thermal Rayleigh number

$$R_{Ec} = \frac{(\Lambda k^4 + Da^{-1}k_1^2)(k_2^2 - R_I)(4\pi^2 - R_I)}{4\pi^4\alpha^2}, \tag{13}$$

$k^2 = \pi^2(1 + \alpha_c^2)$,
$k_1^2 = \pi^2(\varepsilon^{-1} + \alpha_c^2)$ and
$k_2^2 = \pi^2(1 + \eta\alpha_c^2)$, α is the wave number.

Eq. (13) is the critical value of R_E discussed later in the section. The normal mode solution (12) satisfies the boundary conditions in Eq. (11). In Eq. (12), $\pi\alpha_c$ is the horizontal wave number. The quantities Ψ_0 and Θ_0 are, respectively, amplitudes of the stream function and temperature. Substituting Eq. (12) into the linearized version of Eqs. (9) and (10) and integrating the above equation with respect to X in $\left[0, \frac{2\pi}{\pi\alpha_c}\right]$ and also with respect to Z in [0,1], we obtain a set of homogeneous equations in Ψ_0 and Θ_0.

In obtaining a non-trivial solution for Ψ_0 and Θ_0, we require

$$r_E = \frac{\left[\sigma + \frac{Pr}{k^2}\left(\Lambda k^4 + Da^{-1}k_1^2\right)\right]\left(\sigma + k_2^2 - R_I\right)(4\pi^2 - R_I)}{4Pr\pi^4\alpha^2}. \tag{14}$$

The scaled thermal Rayleigh number r_E is the eigenvalue of the problem that throws light on the stability or otherwise of the system.

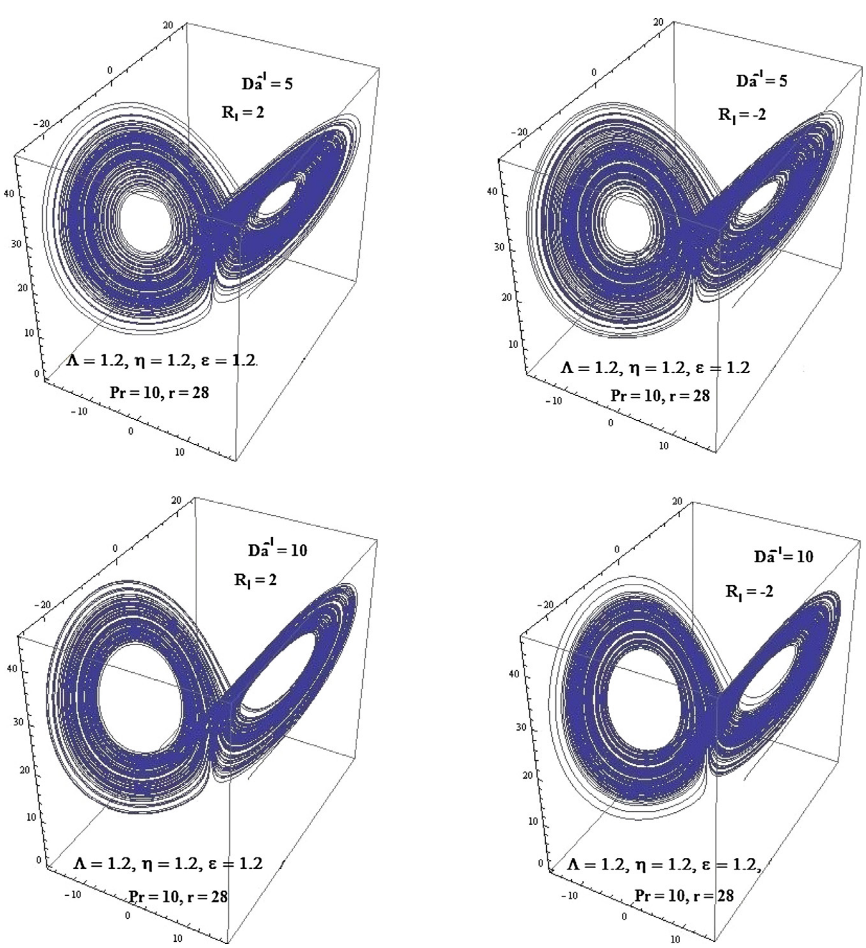

Fig. 2. Phase-space plots involving the amplitudes A, B and C for different values of Da^{-1} and R_I.

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The onset of convection can occur in one of the following ways:

(i) stationary convection or
(ii) oscillatory convection.

The critical value of R_E, i.e., R_{Ec} signifies the onset of convection via one of the above modes. R_{Ec} of stationary is generally different from R_{Ec} of oscillatory. If R_{Ec} of stationary convection is less than that of oscillatory convection, then we say the principle of exchange of stabilities (PES) is valid. We now move over to the discussion on the stationary instability followed by that on the validity or otherwise of the PES.

4. Marginal stationary state

If σ in Eq. (12) is real, then the marginal stationary instability occurs when $\sigma = 0$. This gives the stationary thermal Rayleigh number in the form

$$r_E^S = 1. \tag{15}$$

The critical wave number α_c satisfies the equation:

$$2\Lambda\eta\pi^4\alpha_c^6 + \pi^2[\Lambda\pi^2(1+2\eta) + (\eta Da^{-1} - \Lambda R_I)]\alpha_c^4 + (R_I - \pi^2)(\Lambda\pi^2)\left(\Lambda\pi^2 - \frac{Da^{-1}}{\varepsilon}\right) = 0. \tag{16}$$

5. Marginal oscillatory state

Taking $\sigma = i\omega$ (ω being the frequency of oscillations) in Eq. (14) and separating the real and imaginary parts, we obtain the scaled, oscillatory thermal Rayleigh number, r_E^0, in the form:

$$r_E^0 = 1 - \frac{\omega^2 k^2}{Pr\left(\Lambda k^4 + Da^{-1}k_1^2\right)\left(k_2^2 - R_I\right)} + i\omega N, \tag{17}$$

where

$$N = \frac{k^2\left(k_2^2 - R_I\right) + Pr\left(\Lambda k^4 - Da^{-1}k_1^2\right)}{Pr\left(k_2^2 - R_I\right)\left(\Lambda k^4 - Da^{-1}k_1^2\right)}.$$

Since r_E^0 is a real quantity, the imaginary part of Eq. (17) has to vanish. This gives rise to two possibilities:

(i) $\omega \neq 0, N = 0$ (oscillatory instability) or
(ii) $\omega = 0, N \neq 0$ (stationary instability).

Taking $N = 0$, we get $k^2(k_2^2 - R_I) + Pr(\Lambda k^4 + Da^{-1}k_1^2) = 0$, which is independent of ω. In problems wherein oscillatory convection is also possible, the condition $N = 0$ leads to an expression for ω^2 that is in turn substituted in the real part of the expression for r_E^0, thereby yielding the scaled, oscillatory thermal Rayleigh number. In view of the fact that N is independent of ω, we infer that

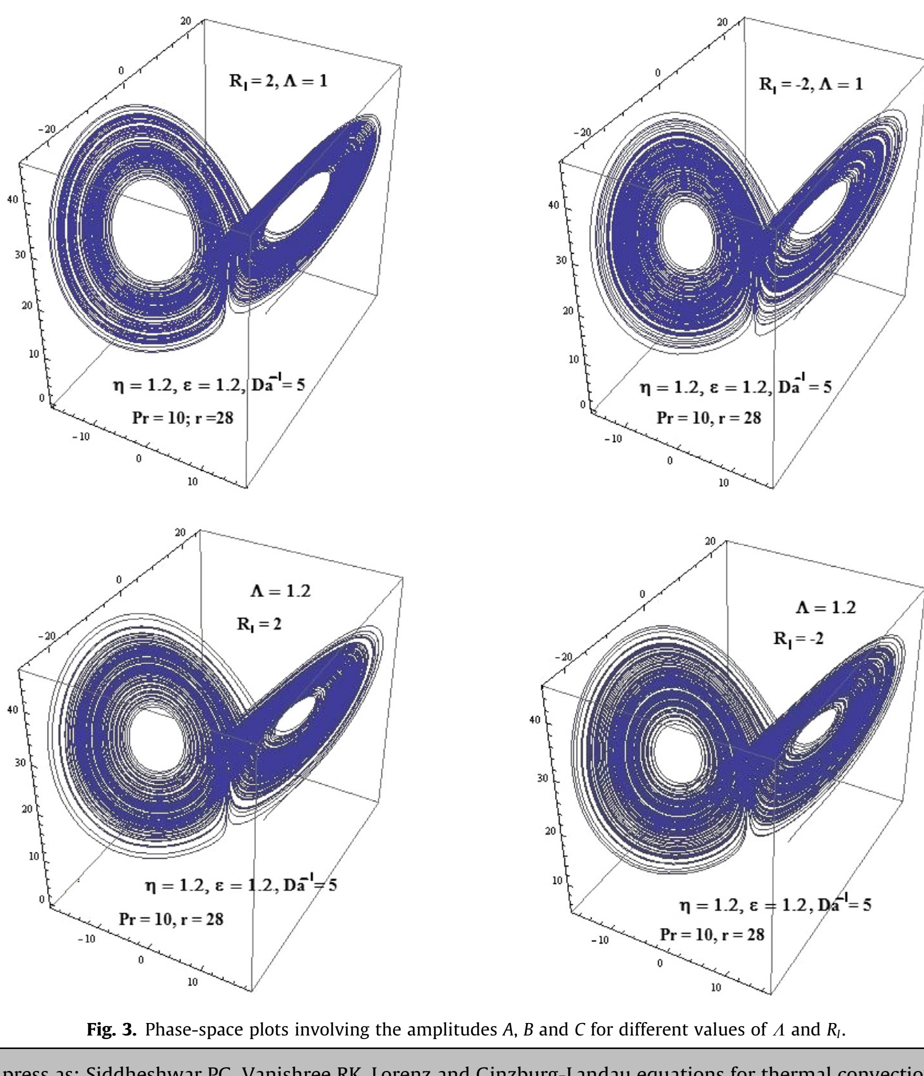

Fig. 3. Phase-space plots involving the amplitudes A, B and C for different values of Λ and R_I.

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oscillatory convection is not possible in the present problem. This essentially means that the PES holds good for the problem at hand.

The linear theory discussed in the previous section reveals that stationary convection is the only possible mode of instability and that oscillatory mode can be discounted. The linear theory predicts only the condition for the onset of convection and is silent about the heat transport. We now embark on a weakly non-linear analysis by means of a truncated representation of Fourier series for velocity and temperature fields to find the effect of various parameters on finite amplitude convection and to know the amount of heat transfer. Specifically we consider the most minimal mode for studying nonlinear instability. We note that the results obtained from such an analysis can serve as starting values while solving a more general nonlinear convection problem.

6. Local non-linear stability theory

The first effect of nonlinearity is to distort the temperature field through the interaction of Ψ and Θ. The distortion of temperature field will correspond to a change in the horizontal mean, i.e., a component of the form $\sin(2\pi z)$ will be generated. Thus a minimal double Fourier series which describes the finite amplitude convection in a porous medium is

$$\left.\begin{aligned}\Psi(X,Z,\tau) &= \frac{\sqrt{2}}{\pi^2\alpha_c}\left(k_2^2 - R_I\right)A(\tau)\sin\pi\alpha X\sin\pi Z\\ \Theta(X,Z,\tau) &= \frac{4\pi}{r_E\left(4\pi^2 - R_I\right)}\left[\sqrt{2}B(\tau)\cos\pi\alpha X\sin\pi Z - C(\tau)\sin 2\pi Z\right]\end{aligned}\right\}, \tag{18}$$

where A is the amplitude of velocity convection, B and C are the amplitudes of temperature to be determined from the dynamics of the system. Substituting Eq. (18) into Eqs. (9) and (10) and following standard orthogonalization procedure for the Galerkin expansion, we obtain the following nonlinear autonomous system (generalized Lorenz model, [38]) of differential equations:

$$\dot{A} = aPr(B - A), \tag{19}$$

$$\dot{B} = k^{-2}(1 - R_I k_2^{-2})[r_E A - B - AC], \tag{20}$$

$$\dot{C} = k^{-2}(1 - R_I k_2^{-2})[AB - bC], \tag{21}$$

where $a = \Lambda + Da^{-1}k_1^2k^{-4}$, $b = (4\pi^2k_2^{-2} - R_Ik_2^{-2})(1 - R_Ik_2^{-2})^{-1}$ and the over dot denotes time derivative with respect to τ. It is important to observe that the nonlinearities in Eqs. (19)–(21) stem from the convective terms in the energy Eq. (3) as in the classical Lorenz system [21].

More models other than minimal ones have not been considered in the study in view of the observation by Siddheshwar and Titus [35] that additional modes do not significantly alter the results on onset of convection as well as heat transport.

7. Steady finite-amplitude convection

Having made a qualitative analysis of the linear autonomous system, we note that the nonlinear system of autonomous differential Eqs. (19)–(21) is not amenable to analytical treatment for

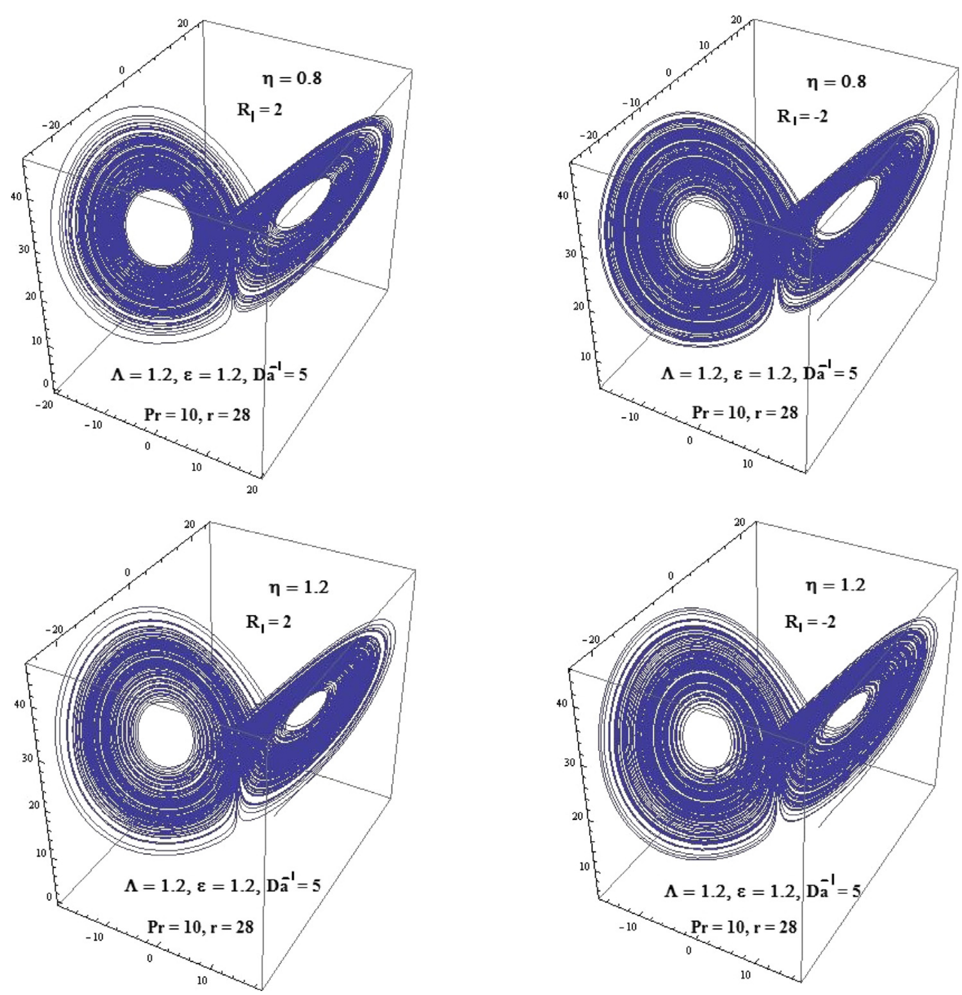

Fig. 4. Phase-space plots involving the amplitudes A, B and C for different values of η and R_I.

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the general time-dependent variables and we need to solve it by means of a numerical method. However, in the case of steady motions, these equations can be solved in closed form. Such solutions prove very useful because they may show that a finite amplitude steady solution to the system is possible for marginal stability. Thus in the case of steady motions, Eqs. (19)–(21) take the form

$$-aPrA + aPrB = 0, \tag{22}$$

$$r_E A - B - AC = 0, \tag{23}$$

$$AB - bC = 0. \tag{24}$$

The solution of system (22)–(24) is,

$$A = \pm\sqrt{b(r_E - 1)}, \quad B = \pm\sqrt{b(r_E - 1)}, \quad C = r_E - 1 \tag{25}$$

We take the positive sign in front of the square root in Eq. (25) on the ground that the amplitude of the stream function is real.

8. Heat transport

The horizontally-averaged Nusselt number, Nu, for the stationary mode of convection (the preferred mode in this problem) is given by

$$Nu(\tau) = \frac{\left[\frac{\alpha_c}{2}\int_{X=0}^{2/\alpha_c}[(1-Z)+\Theta]_{,Z}dX\right]_{Z=0}}{\left[\frac{\alpha_c}{2}\int_{X=0}^{2/\alpha_c}(1-Z)_{,Z}dX\right]_{Z=0}}. \tag{26}$$

Substituting Eq. (18) in Eq. (26) and completing the integration, we get

$$Nu(\tau) = 1 + 2\left[r_E\left(1 - \frac{R_I}{4\pi^2}\right)\right]^{-1} C(\tau). \tag{27}$$

The second term on the right side of Eq. (27) characterizes the convective contribution to the heat transport. To quantify the Nusselt number we need to know $C(\tau)$ and hence there is a need to solve the Lorenz system Eqs. (19)–(21). Alternately we may write $B(\tau)$ and $C(\tau)$ in terms of $A(\tau)$ and arrive at the Ginzburg-Landau model. This is done in the succeeding section.

9. The Ginzburg-Landau equation from the Lorenz model

From Eqs. (19) and (20), we have

$$B = [aPr]^{-1}\dot{A} + A \quad \text{and} \tag{28}$$

$$C = \frac{1}{A}\left[r_E A - B - \frac{1}{k^{-2}\left(k_2^2 - R_I\right)}\dot{B}\right]. \tag{29}$$

Using Eq. (28) in Eq. (29), we get

$$C = \frac{1}{A}[D_1 A - D_2\dot{A} - D_3\ddot{A}], \tag{30}$$

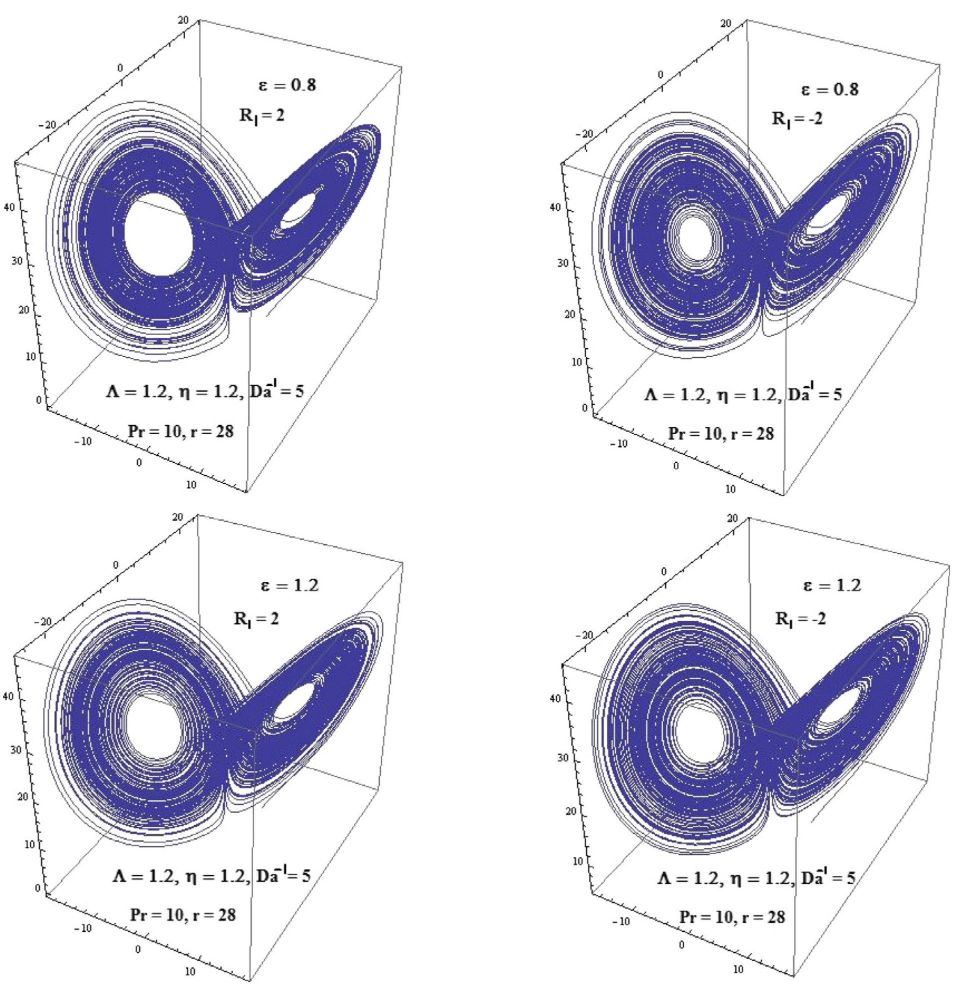

Fig. 5. Phase-space plots involving the amplitudes A, B and C for different values of ϵ and R_I.

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where

$$D_1 = r_E - 1, D_2 = [aPr]^{-1} + k^2\left(k_2^2 - R_I\right)^{-1} \text{ and}$$
$$D_3 = k^2\left[aPr\left(k_2^2 - R_I\right)\right]^{-1}.$$

Substituting Eqs. (28) and (30) in Eq. (21) we get a third order equation in A.

$$D_2\dot{A}^2 + D_3\dot{A}\ddot{A} - D_2A\ddot{A} - D_3A\dddot{A} = k^{-2}(1 - R_Ik_2^-2[(aPr)^{-1}A^3\dot{A} + A^4 - bD_1A^2 + bD_2\dot{A} + D_3\ddot{A}], \quad (31)$$

Neglecting terms of type $\left(\frac{dA}{d\tau}\right)^2, \frac{dA}{d\tau}\frac{d^2A}{d\tau^2}, A^2\frac{dA}{d\tau}, A\frac{d^2A}{d\tau^2}$ and $A\frac{d^3A}{d\tau^3}$, we get the following equation:

$$bD_2\frac{dA}{d\tau} = bD_1A - A^3. \quad (32)$$

Eq. (32) is obviously the Ginzburg-Landau model for non-linear convection in a Newtonian fluid-saturated anisotropic porous medium with heat source. Substituting Eq. (32) in Eq. (30), we get $C(\tau)$ in terms of $A(\tau)$ in the form:

$$C(\tau) = \frac{-D_1^2D_3}{D_2^2} + \frac{1}{b}\left(1 + \frac{D_1D_3[3b+1]}{D_2^2}\right)A^2 - \frac{3D_3}{b^2D_2^2}A^4. \quad (33)$$

Solving Eq. (32) for $A(\tau)$, we get

$$A(\tau) = \left[\frac{1}{bD_1} + \left(\frac{1}{A(0)^2} - \frac{1}{bD_1}\right)e^{-\frac{2D_1}{D_2}\tau}\right]^{-1/2}. \quad (34)$$

where $A(0)$ is the initial amplitude. It is apparent from the above that Eq. (27) is an analytical expression for the Nusselt number with $C(\tau)$ given by Eq. (33).

10. Results and discussion

Heat transport by thermal convection in a porous medium with heat source(sink) is investigated using a minimal Fourier series. The controlling parameter is the external Rayleigh number, R_E, which is influenced by the heat source(sink) parameter, R_I, (internal Rayleigh number). These two Rayleigh numbers arise in the study due to external heating (R_E) and internal heat generation (R_I). In view of the fact that we intend to study buoyancy induced convection, we assume that R_I is small enough not to induce convection by itself. Hence we have assumed small positive values of R_I to represent a heat source and small negative values of R_I for a heat sink so that it only influences in a weak sense, the critical value of R_E. Onset of convection and heat transport are also influenced by inverse Darcy number, Da^{-1}, Brinkman number, Λ, and mechanical and thermal anisotropy parameters ε and η. Da^{-1} represents the structure of the porous medium, ε and η represent the differential packing of the spherical particles in the horizontal and vertical directions. It is the mechanical anisotropy that leads to thermal anisotropy. The porous medium under consideration is a sparsely packed one and hence we assume Da^{-1} to take the values from 1 to 1000. The quantity Λ can assume a range of values that are greater than, equal to or less than 1 (see Givler and Altobelli [11]), and ε and η are assumed to take values in a neighbourhood involving unity. The values of ε and η greater than 1 would indicate that the horizontal medium or fluid properties are relatively greater than their vertical counterparts. PES is valid and hence stationary convection is the preferred mode. It is in this sense that slow time scale is considered. Theoretical analysis shows that subcritical instability is possible which thus discounts supercritical instability (see Fig.1).

The minimal representation of Fourier series gave us the Lorenz model for high-porosity media using which we arrived at the Ginzburg-Landau Eq. (32) of the problem. This equation is a Ber-

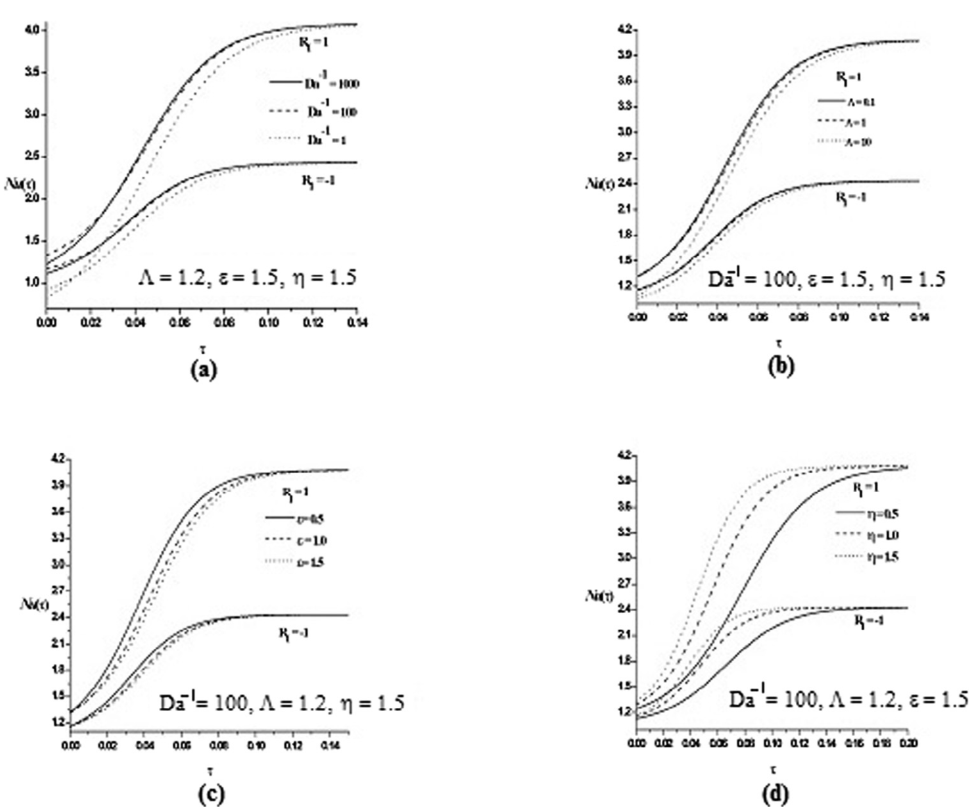

Fig. 6. Plot of Nu versus τ for heat source and sink for different values of: (a) Da^{-1} for fixed Λ, ϵ and η, (b) Λ for fixed ϵ, η and Da^{-1}, (c) ϵ for fixed Λ, η and Da^{-1}, (d) η for fixed Λ, ϵ and Da^{-1}.

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Table 1
Limiting cases of the present study (D_2 and D_3 are coefficients appearing in the Ginzburg-Landau Eq. (32)).

k_1^2	k_2^2	$\Lambda k^2 + Da^{-1}$	Type of convection (Isotropic medium)	Ra_c	D_2	D_3
k^2	k^2	Da^{-1}	Bénard–Darcy			
			(i) With heat source	$\frac{Da^{-1}k^2(k^2-R_I)(4\pi^2-R_I)}{4\pi^4\alpha^2}$	$\frac{k^2}{PrDa^{-1}}+\frac{k^2}{k^2-R_I}$	$\frac{k^4}{PrDa^{-1}(k^2-R_I)}$
			(ii) Without heat source [32–34,36]	$\frac{Da^{-1}k^4}{\pi^2\alpha^2}$	$1+\frac{k^2}{PrDa^{-1}}$	$\frac{k^2}{PrDa^{-1}}$
k^2	k^2	k^2	Bénard–Rayleigh			
			(i) With heat source	$\frac{k^4(k^2-R_I)(4\pi^2-R_I)}{4\pi^4\alpha^2}$	$\frac{1}{Pr}+1\,\frac{k^2}{k^2-R_I}$	$1+\frac{1}{Pr}$
			(ii) Without heat source [31]	$\frac{k^6}{\pi^2\alpha^2}$	$\frac{k^2}{Pr(k^2-R_I)}$	$\frac{1}{Pr}$
k^2	k^2	k^2+Q	Bénard–Chandrasekhar			
			(i) With heat source	$\frac{(k^2+Q)k^2(k^2-R_I)(4\pi^2-R_I)}{4\pi^4\alpha^2}$	$\frac{k^2}{Pr(k^2+Q)}+\frac{k^2}{k^2-R_I}$	$\frac{k^4}{Pr(k^2+Q)(k^2-R_I)}$
			(ii) Without heat source [32–34,36]	$\frac{(k^2+Q)k^4}{\pi^2\alpha^2}$	$1+\frac{k^2}{Pr(k^2+Q)}$	$\frac{k^2}{Pr(k^2+Q)}$

noulli differential equation and has an analytical solution (34). Starting from an assumed initial amplitude $A(0)$ we can obtain the amplitude $A(\tau)$ and thereby the other amplitudes $B(\tau)$ and $C(\tau)$. The amplitude $C(\tau)$ is then used to quantify the heat transport in terms of the Nusselt number, Nu. The results of extensive computation are shown in Figs. 2–6. Figs. 2–5 are the phase-space plots involving the amplitudes A, B and C for different values of $R_I, Da^{-1}, \eta, \epsilon$ and Λ. Fig. 6 is a plot of the Nusselt number versus time, τ for various parameters' values.

Fig. 2 is the phase space plot for different values of Da^{-1} and R_I. The effect of Da^{-1} for both $R_I > 0$ and $R_I < 0$ is to open out the trajectories and move towards the edge of finite bounds of the trajectories.

The effect of η on the trajectories is depicted in Fig. 3. It is seen that the effect of η for both $R_I > 0$ and $R_I < 0$ is to move trajectories towards the critical points.

From Fig. 4 it is observed that the effect of ϵ in the presence of a heat source is to move the trajectories inwards and towards the critical points where as for a sink the trajectories move away from the critical points.

The effect of Λ on the trajectories in the presence of both heat source and sink is to move the trajectories closer to the critical points.

From Figs. 2–5, it can be seen that the trajectories move out or away from the critical points for $R_I > 0$ (source) due to gain in the energy and close in for $R_I < 0$ (sink) due to removal of energy.

From Fig. 6 it is clear that heat source ($R_I > 0$) enhances the heat transport and heat sink ($R_I < 0$) diminishes it. The effect of increasing the value of the inverse Darcy number, Da^{-1}, in the case of both heat source and heat sink, is to augment the heat transport.

From Fig. 6(a) it is also clear that Da^{-1} enhances the heat transport in the range $Da^{-1} = 1$ to 100. But as Da^{-1} increases further (ie., say $Da^{-1} = 1000$), heat transfer diminishes initially and after a short time it increases. Fig. 6(b) is a plot of Nu versus τ for different values of Λ. The effects of ε and η on Nu is opposite to each other which is seen in Fig. 6(c) and (d).

Several limiting cases are obtained from the present study and these are indicated in Table 1. In the absence of heat source we get the results of the classical Bénard-Darcy, Bénard-Rayleigh and Bénard-Chandrasekhar problems (reported in Siddheshwar et al. [31]).

11. Conclusions

- It is advantageous to use Ginzburg Landau equation as it facilitates an exact representation for the amplitudes as well as Nusselt number, Nu.
- The general formulation of the Brinkman-Bénard problem covers Bénard-Darcy, Bénard-Rayleigh and Bénard-Chandrasekhar convection problems, as limiting cases (see Table 1).

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